ME 6135: Advanced Aerodynamics

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Continuity Equation for steady two-dimensional flows (in differential form):

$$\nabla \cdot (\rho \vec{V}) = 0 \qquad \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

for compressible flow

$$\nabla \cdot \vec{V} = 0 \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

for incompressible flow

Condition of irrotationality in case of two-dimensional flows: (**curl of velocity =0)**

$$\text{curl } \vec{\mathbf{V}} = 0 \ (\nabla \times \vec{\mathbf{V}} = 0) \qquad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

Stream function, ψ

Stream function, ψ (x, y, t) is a single function by which the two entities of velocity components u (x, y, t) and v (x, y, t) of a two-dimensional incompressible flow can be defined. Consider continuity equation-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Define ***stream function*** by the following definition (for incompressible flow) -

$$u \equiv \frac{\partial \psi}{\partial y} \quad \text{and} \quad v \equiv -\frac{\partial \psi}{\partial x}$$

$$v_r \equiv \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta \equiv -\frac{\partial \psi}{\partial r} \quad (r, \theta \,\text{coordinate})$$

This definition automatically satisfy the ***continuity equation*** as-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

Thus, **stream function** is a single function which **satisfy** the first governing equation in fluid dynamics i.e. the **continuity equation**.

Stream function, ψ

Continuity equation for two-dimensional compressible flow -

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

Define ***stream function*** by the following definition (for compressible flow) -

$$u \equiv \frac{1}{\rho}\frac{\partial \psi}{\partial y} \quad \text{and} \quad v \equiv -\frac{1}{\rho}\frac{\partial \psi}{\partial x}$$

$$v_r \equiv \frac{1}{\rho}\frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta \equiv -\frac{1}{\rho}\frac{\partial \psi}{\partial r}(r, \theta \text{coordinate})$$

This definition automatically satisfy the ***continuity equation*** as-

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = \frac{\partial}{\partial x}\left(\rho \times \frac{1}{\rho}\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\rho \times \frac{1}{\rho}\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

Thus, **stream function** is a single function which **satisfy** the first governing equation in fluid dynamics i.e. the **continuity equation**.

Velocity Potential, φ

Velocity Potential, Φ (x, y, t) is another function by which the two entities of velocity components u (x, y, t) and v (x, y, t) of a two-dimensional irrotational incompressible flow can be defined.
Consider the **condition of irrotationality**

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

Define ***velocity potential*** by the following definition-

$$u \equiv \frac{\partial \phi}{\partial x} \quad \text{and} \quad v \equiv \frac{\partial \phi}{\partial y}$$

$$v_r \equiv \frac{\partial \phi}{\partial r} \quad \text{and} \quad v_\theta \equiv \frac{1}{r}\frac{\partial \phi}{\partial \theta} \quad (r, \theta \, \text{coordinate})$$

This definition automatically satisfy the ***condition of irrotationality*** as-

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} = 0$$

Thus, **velocity potential** is a function which **satisfy the *condition of irrotationality*** .

Relation between ϕ and ψ

It can be seen that-

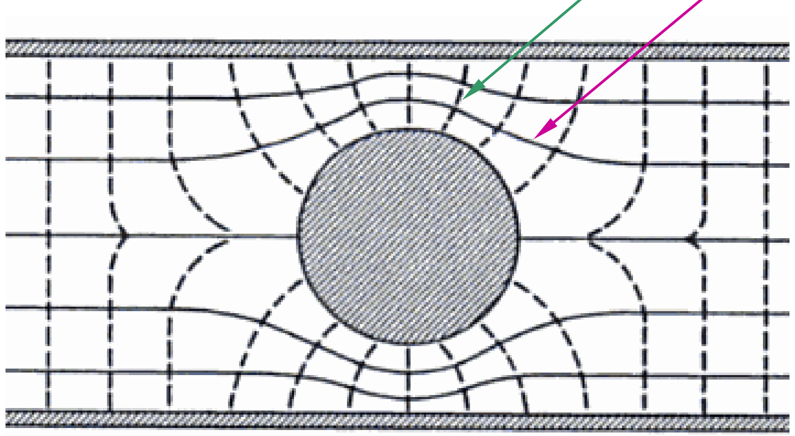
$$\left(\frac{dy}{dx}\right)_{\psi=\text{constant}} = \frac{v}{u}$$ **streamline**

$$\left(\frac{dy}{dx}\right)_{\phi=\text{constant}} = -\frac{u}{v}$$ **equipotential line**

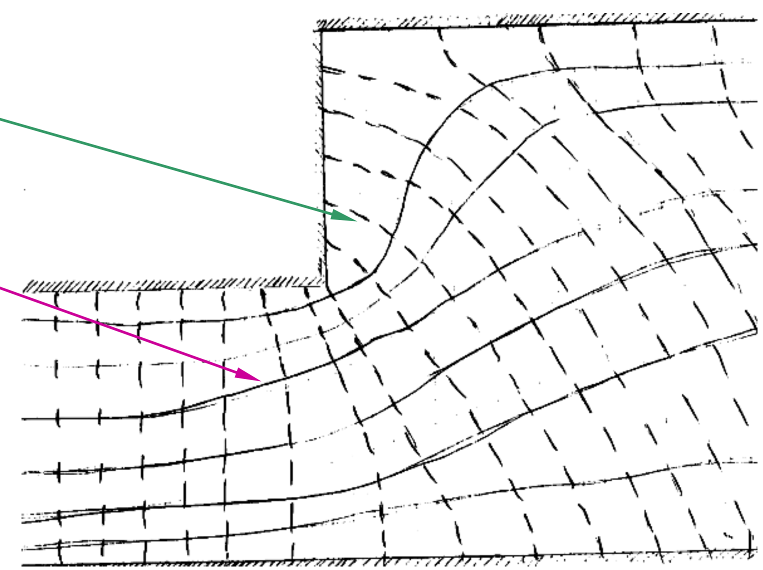
Streamlines and equipotential lines are mutually perpendicular.

Velocity potential

streamline



Flow net



Flow net

Laplace Equation

Consider 2D irrotational, incompressible flow: the velocity components can be defined in terms of both the stream function and velocity potential-

$$u = \frac{\partial \psi}{\partial y} \; ; \; v = -\frac{\partial \psi}{\partial x}$$

$$u = \frac{\partial \phi}{\partial x} \; ; \; v = \frac{\partial \phi}{\partial y}$$

Now use the expression of **stream function in the condition of irrotationality**:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0$$

$$\Rightarrow - \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\Rightarrow \nabla^2 \psi = 0$$

Laplace Equation

Similarly, use the expression of **velocity potential in the continuity equation**:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right)=0$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0$$

$$\Rightarrow \nabla^2 \phi = 0$$

$$\nabla^2 \psi = 0$$

$$\nabla^2 \phi = 0$$

***Ψ* and *Φ* both satisfy the Laplace equation**

The equations of stream function and velocity potential are in the forms of Laplace's equation- an equation that arise in many areas of physical sciences and engineering.

The functions ψ and Φ that satisfy the **Laplace's equation** represents a possible two-dimensional, incompressible, inviscid, irrotational flow field i.e. the **Potential flow**.

Problem

Consider a flow field given by

$$\psi = 3(x^2 - y^2)$$

Show that the flow is irrotational. Determine the velocity potential for this flow.

Solution:

$$x - \text{velocity component}, u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(3x^2 - 3y^2) = -6y$$

$$y - \text{velocity component}, v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(3x^2 - 3y^2) = -6x$$

$$\text{angular velocity}, \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\Rightarrow \omega_z = \frac{1}{2}\left(\frac{\partial(6x)}{\partial x} - \frac{\partial(6y)}{\partial y}\right)$$

$$\Rightarrow \omega_z = \frac{1}{2}(6-6) = 0$$

So, the flow is irrotational.

Another approach:

$$\text{If the flow is irrotational; Laplace equation}, \nabla^2 \psi = 0$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

Since the flow is irrotational, there must exist a velocity potential for this flow.

Again, from the definition of velocity potential,

$$x-\text{velocity component}, u=\frac{\partial\phi}{\partial x}=-6y$$

$$\Rightarrow \phi=\int -6y\,dx+f(y) \qquad ; \quad f(y)\,\text{is an arbitrary function of } y$$

$$\Rightarrow \phi=-6xy+f(y)$$

$$y-\text{velocity component}, v=\frac{\partial\phi}{\partial y}=-6x$$

$$\Rightarrow \frac{\partial}{\partial y}\left(-6xy+f(y)\right)=-6x$$

Φ from earlier

$$\Rightarrow -6x+\frac{df(y)}{dy}=-6x$$

$$\Rightarrow \frac{df(y)}{dy}=0 \qquad \therefore \; f=\text{constant}$$

$$\therefore \; \phi=-6xy+\text{constant}$$

Since ϕ and ψ are used to determine the velocity components by differentiation, the constant is of no concern; it is usually set to zero. Hence

$$\boxed{\phi = -6xy}$$

$$\psi = 3(x^2 - y^2) \qquad \therefore\ d\psi = 6xdx - 6ydy = 0 \quad \text{at}\ \psi = \text{C}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\psi=\text{C}} = \frac{x}{y}$$

$$\phi = -6xy \qquad \therefore\ d\phi = -6xdy - 6ydx = 0 \quad \text{at}\ \phi = \text{C}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\phi=\text{C}} = -\frac{y}{x}$$

$$\therefore\ \left.\frac{dy}{dx}\right|_{\psi=\text{C}} \times \left.\frac{dy}{dx}\right|_{\phi=\text{C}} = \frac{x}{y} \times -\frac{y}{x} = -1$$

Therefore lines of constant ϕ are orthogonal to lines of constant ψ.

Problem

The velocity in a flow field is given by

$$\vec{\mathbf{V}} = (x^2 y - xy^2)\ \hat{i} + \left(\frac{y^3}{3} - xy^2\right)\ \hat{j}$$

(a) Does a stream function exist? If a stream function exists, what is it?
(b) Does a potential function exist? If a potential function exists, what is it?

Problem

A velocity field is proposed to be

$$\vec{V} = \frac{10y}{x^2+y^2}\,\hat{i} - \frac{10x}{x^2+y^2}\,\hat{j}$$

(a) Is this a possible incompressible flow?
(b) If so, find the pressure gradient with z-axis vertical. Use ρ =1.23 kg/m^3 and consider the fluid is frictionless.

Solution

(a) The differential continuity equation is to be checked to determine if the velocity field is possible or not.

$$u = \frac{10y}{x^2+y^2} \qquad \therefore \frac{\partial u}{\partial x} = \frac{-20xy}{(x^2+y^2)^2}$$

$$v = -\frac{10x}{x^2+y^2} \qquad \therefore \frac{\partial v}{\partial y} = \frac{20xy}{(x^2+y^2)^2}$$

$$\text{Now, } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{-20xy}{(x^2+y^2)^2} + \frac{20xy}{(x^2+y^2)^2} = 0$$

Since the given velocity field satisfies the continuity equation, thus this field represents a possible incompressible flow.

Problem

(b) Consider Euler equation (for frictionless fluid)

$$x-\text{momentum}:\quad \rho\frac{Du}{Dt}=\rho f_x-\frac{\partial p}{\partial x}$$

$$\Rightarrow \rho\left(\cancel{\frac{\partial u}{\partial t}}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\cancel{w}\frac{\partial u}{\partial z}\right)=\cancel{\rho f_x}-\frac{\partial p}{\partial x}$$

; steady 2D flow, *z*-axis vertical

$$\Rightarrow \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}$$

$$\Rightarrow \frac{\partial p}{\partial x}=\frac{123x}{(x^2+y^2)^2}$$

$$y-\text{momentum}:\quad \rho\frac{Dv}{Dt}=\rho f_y-\frac{\partial p}{\partial y}$$

$$\Rightarrow \rho\left(\cancel{\frac{\partial v}{\partial t}}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\cancel{w}\frac{\partial v}{\partial z}\right)=\cancel{\rho f_y}-\frac{\partial p}{\partial y}$$

; steady 2D flow, *z*-axis vertical

$$\Rightarrow \rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}$$

$$\Rightarrow \frac{\partial p}{\partial y}=\frac{123y}{(x^2+y^2)^2}$$

Problem

$$z-\text{momentum}: \quad \rho\frac{Dw}{Dt} = \rho f_z - \frac{\partial p}{\partial z}$$

; steady 2D flow, z-axis vertical, $f_z = -g = -\ 9.81\text{m/s}^2$

$$\Rightarrow 0 = \rho(-g) - \frac{\partial p}{\partial z}$$

$$\Rightarrow \frac{\partial p}{\partial z} = (1.23)(-9.81) = -12.07$$

So, the pressure gradient:

$$\nabla p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$$

$$\boxed{\nabla p = \frac{123}{(x^2+y^2)^2}(x\hat{i} + y\hat{j}) - 12.07\hat{k}}$$

Simple problems to be solved-

1. Determination of stream function and velocity potential
2. Confirmation of possible potential flow etc.

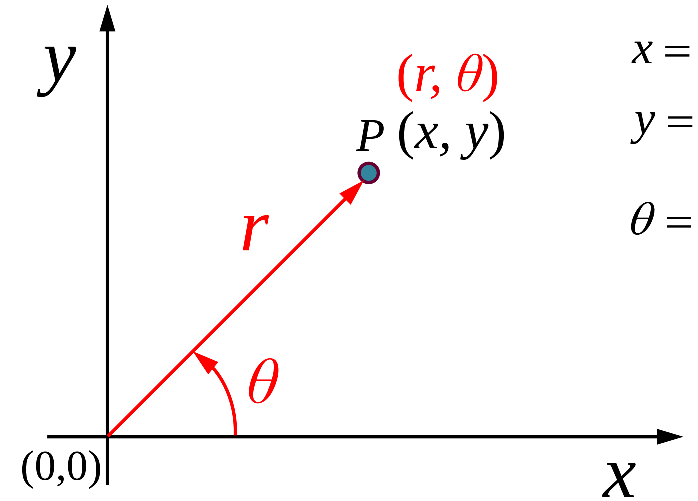
Coordinate systems

To apply the governing equations, a coordinate system: (x,y,z) or (r,θ,z) is to be chosen that best fits the geometry of the flow problem to be solved.

The velocity field can be expressed by :

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$ (Cartesian (x,y,z))

$$\vec{V} = v_r\hat{i}_r + v_\theta\hat{i}_\theta + v_z\hat{i}_z$$ (Cylindrical (r,θ,z))



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

2D coordinate system

The “vector” or “del” operator has the following two forms depending on coordinate system:

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$ (Cartesian (x,y,z))

$$\nabla = \hat{i}_r\frac{\partial}{\partial r} + \hat{i}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{i}_z\frac{\partial}{\partial z}$$ (Cylindrical (r,θ,z))

Continuity equation for steady inviscid incompressible flows:

$$\boxed{\nabla \cdot \vec{V} = 0}$$

(divergence free velocity field)

For 2D flows:

$$\nabla \cdot \vec{V} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} \right) \cdot \left(u\hat{i} + v\hat{j} \right) = 0$$

$$\boxed{\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

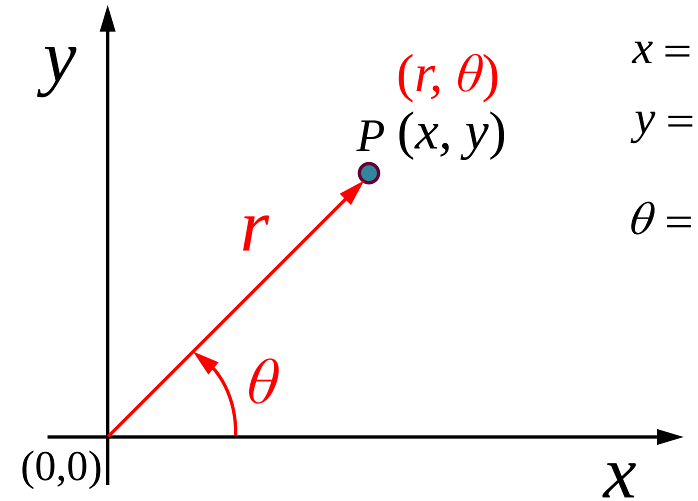
(Cartesian (x,y))

$$\nabla \cdot \vec{V} = \left(\hat{i}_r \frac{\partial}{\partial r} + \hat{i}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} \right) \cdot \left(v_r \hat{i}_r + v_\theta \hat{i}_\theta \right) = 0$$

$$\Rightarrow \frac{1}{r} \left(\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_\theta}{\partial \theta} \right) = 0$$

$$\boxed{\Rightarrow \frac{\partial (r v_r)}{\partial r} + \frac{\partial v_\theta}{\partial \theta} = 0}$$

(Cylindrical (r,θ))



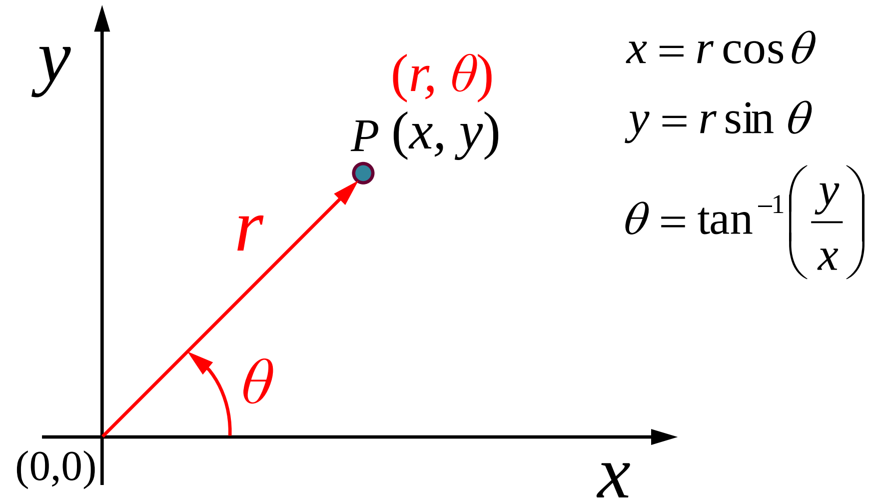
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

2D coordinate system

Stream function, ψ in (r,θ)

For 2D flows in polar coordinate (r, θ) continuity equation:

$$\Rightarrow \frac{\partial(rv_r)}{\partial r} + \frac{\partial v_\theta}{\partial \theta} = 0$$



2D coordinate system

Define ***stream function*** by the following definition-

$$v_r \equiv \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$ and $$v_\theta \equiv -\frac{\partial \psi}{\partial r}$$

$$\therefore \quad \frac{\partial(rv_r)}{\partial r} = \frac{\partial}{\partial r}\left(r\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right) = \frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial \theta}\right) = \frac{\partial^2 \psi}{\partial r \partial \theta}$$

and $\therefore \quad \dfrac{\partial v_\theta}{\partial \theta} = \dfrac{\partial}{\partial \theta}\left(-\dfrac{\partial \psi}{\partial r}\right) = -\dfrac{\partial^2 \psi}{\partial r \partial \theta}$

Now $\quad \dfrac{\partial(rv_r)}{\partial r} + \dfrac{\partial v_\theta}{\partial \theta} = \dfrac{\partial^2 \psi}{\partial r \partial \theta} - \dfrac{\partial^2 \psi}{\partial r \partial \theta} = 0 \quad$ which **satisfy** the **continuity equation**.

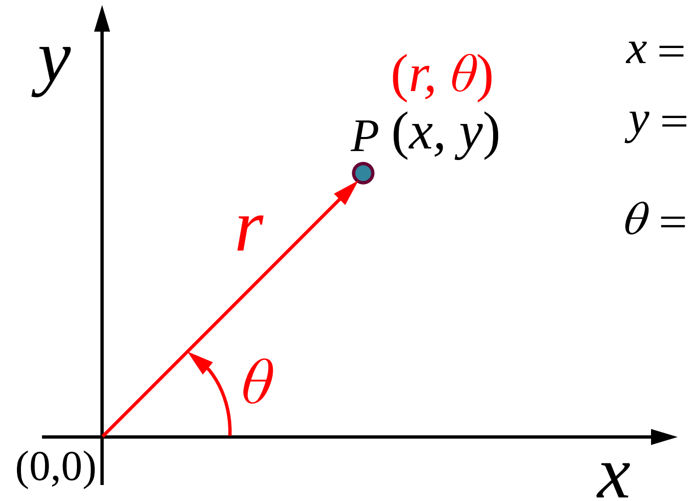
Condition of irrotationality for steady inviscid flows:

$$\boxed{\nabla \times \vec{V} = 0}$$ (Curl of velocity field is zero)

For 2D flows:

$$\nabla \times \vec{V} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} \right) \times \left(u\hat{i} + v\hat{j} \right) = 0$$

$$\boxed{\Rightarrow \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0}$$ (Cartesian (x,y))



$x = r\cos\theta$

$y = r\sin\theta$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

2D coordinate system

$$\nabla \times \vec{V} = \left(\hat{i}_r \frac{\partial}{\partial r} + \hat{i}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \right) \times \left(v_r \hat{i}_r + v_\theta \hat{i}_\theta \right) = 0$$

$$\Rightarrow \frac{\partial v_\theta}{\partial r} + \frac{1}{r}\left(v_\theta - \frac{\partial v_r}{\partial \theta} \right) = 0$$

$$\boxed{\Rightarrow \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} = 0}$$ (Cylindrical (r,θ))

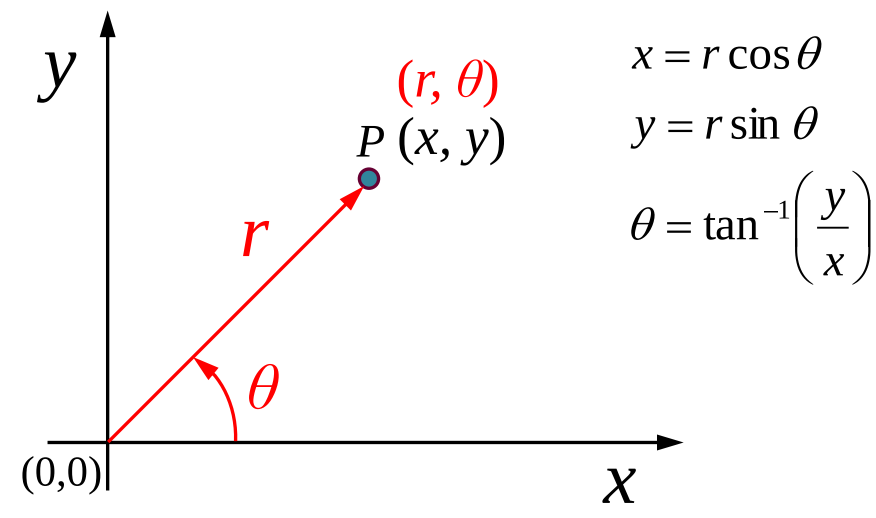
Potential function, φ in (r,θ)

For 2D flows in polar coordinate (r, θ) the **condition of irrotationality:**

$$\Rightarrow \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} = 0$$

Define ***potential function*** by the following definition-

$$v_r \equiv \frac{\partial \varphi}{\partial r} \quad \text{and} \quad v_\theta \equiv \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$



2D coordinate system

$$\therefore \frac{\partial v_r}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{\partial \phi}{\partial r}\right) = \frac{\partial^2 \phi}{\partial r \partial \theta}$$

and $$\frac{\partial v_\theta}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = -\frac{\partial \phi}{\partial \theta}\frac{1}{r^2} + \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} = -(v_\theta r)\frac{1}{r^2} + \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} = -\frac{v_\theta}{r} + \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}$$

Now $$\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} = \left(-\frac{v_\theta}{r} + \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}\right) + \frac{v_\theta}{r} - \frac{1}{r}\left(\frac{\partial^2 \phi}{\partial r \partial \theta}\right) = 0$$

which **satisfy** the **condition of irrotationality**

Problem

A velocity field is given in polar coordinates for a perfect fluid flow as:

$$v_r = \left(\frac{\theta^2}{r} - 1\right) \quad \text{and} \quad v_\theta = (\theta - 2r)$$

Find the stream function for this flow.

Solution:

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad = \left(\frac{\theta^2}{r} - 1\right)$$

$$\Rightarrow \frac{\partial \psi}{\partial \theta} \quad = (\theta^2 - r)$$

$$\therefore \ \psi = \frac{\theta^3}{3} - r\theta + f(r)$$

$$v_\theta = -\frac{\partial \psi}{\partial r} \quad = (\theta - 2r) \ \text{(given)}$$

$$\Rightarrow -\frac{\partial}{\partial r}\left[\frac{\theta^3}{3} - r\theta + f(r)\right] \quad = (\theta - 2r)$$

$$\Rightarrow \theta - \frac{df(r)}{dr} \quad = (\theta - 2r)$$

$$\Rightarrow -\frac{df(r)}{dr} \quad = -2r$$

$$\therefore \ f(r) = r^2 + \text{constant}$$

$$\therefore \ \psi = \frac{\theta^3}{3} - r\theta + r^2 + \text{constant}$$

Problem

A physically possible irrotational flow is:

$$\vec{V} = (2x+1)\hat{i} - (2y)\hat{j}$$

Find the velocity potential function for this flow.

Solution:

$$\therefore \ \phi = x^2 + x - y^2 + \text{constant}$$